ON THE PROPAGATION OF DISTURBANCES IN SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Many oscillatory devices have the so-called wave link-system as the main construction element in which the disturbances propagate with a definite velocity which in a number of cases determines the frequency of the resulting oscillations. Examples of this are the devices known as the hydraulic ram, the vacuum tube generator with a long line load, the violin string, etc. Systems of this kind are described in the simplest cases by equations of the string type; the stationary states of which under known conditions can become unstable with respect to the initial or boundary values. If at the same time the boundary conditions are nonlinear, then in the system there may occur oscillations of a definite amplitude. The corresponding problems for the examples given above have been considered in [1-3]. The results of these investigations have been obtained through reduction of the problems to finite functional equations which can be solved quite descriptively by graphical means with the aid of the Lemerey diagrams. The stability of the resulting periodic solutions was investigated by the method of Koenigs.

The known D'Alembert representation of the general solution is characteristic for the string for which these developments were made. This fact is essential for applicability of the indicated methods. However, even when such representation takes place, the reduction to finite functional equations is possible only for the case when special initial and boundary conditions are given. If the given conditions are more general, then the computations carried out in [1-3] are not valid.

It is possible to indicate the approach for the solution of this more general problem. This approach is not connected with any particular form of the initial or boundary conditions nor with the requirement for the D'Alembert representation of the general solution. The respective developments can be utilized in problems based on the one-dimensional wave equation. The first part of this work is devoted to the presentation of these questions.

The second part of the paper analyses the oscillations in a system with piping and a shut-off valve containing a nonlinear spring. The linear formulation was investigated by Lur'e and Chekmarev [4]. In this example, it is possible to apply the methods of [1-3] because of the special character of the boundary and the initial conditions. By considering the valve mass and friction, the problem is reduced to differential-difference equation of the neutral type.

Here is considered only the case of the weightless valve and no friction. The character of the resulting oscillations is established and their stability investigated.

1. Given the string equation

$$u_{\tau\tau} = u_{xx} \qquad (\tau = at) \tag{1.1}$$

it is required to determine its solution for $\tau > 0$, $x \in [0, l]$ and the following initial and boundary conditions:

$$u(x, 0) = \varphi(x), \qquad u_\tau(x, 0) = \psi(x) \tag{1.2}$$

$$u_x(0, \tau) = F[u(0, \tau)], \qquad u(l, \tau) = 0 \tag{1.3}$$

Here φ, ψ are known functions, F is a given, generally nonlinear function of its argument.

Let us introduce the function $v(x, \tau)$ so that the Equation (1.1) can be replaced by the system

$$u_\tau = v_x, \qquad u_x = v_\tau \tag{1.4}$$

Its solution is given as

$$u = f_1(\tau - x) + f_2(\tau + x), \qquad v = -f_1(\tau - x) + f_2(\tau + x) \tag{1.5}$$

where f_1 and f_2 are functions determined by the additional conditions.

The boundary condition $u(l, \tau) = 0$ gives

$$f_1(\tau - l) = -f_2(\tau + l) \qquad (\tau > 0) \tag{1.6}$$

The first condition (1.3) is expressed as

$$-f_1'(\tau) + f_2'(\tau) = F[f_1(\tau) + f_2(\tau)] \qquad (\tau > 0) \tag{1.7}$$

or, in view of (1.6)

$$f_2'(\tau) + f_2'(\tau + 2l) = F[f_2(\tau) - f_2(\tau + 2l)] \qquad (\tau > 0) \qquad (1.8)$$

We have obtained a nonlinear differential-difference equation of neutral type for determination of the function $f_2(\tau)$. In order that the problem be formulated correctly, it is necessary to prescribe $f_2(\tau)$ for $0 \leqslant \tau \leqslant 2l$. The initial conditions (1.2) easily yield the relationships

$$f_1(-x) + f_2(x) = \varphi(x), \qquad f_1'(-x) + f_2'(x) = \psi(x) \qquad (0 \leqslant x \leqslant l) \qquad (1.9)$$

The prime indicates differentiation with respect to the whole argument. Therefore

$$f_2'(x) = \tfrac{1}{2}[\varphi'(x) + \psi(x)] \qquad (0 \leqslant x \leqslant l) \qquad (1.10)$$

Let us now determine $f_2'(x)$ for $x \in [l, 2l]$. The initial conditions yield the formula

$$f_1'(-x) = \tfrac{1}{2}[\psi(x) - \varphi'(x)] \qquad (0 \leqslant x \leqslant l) \qquad (1.11)$$

Replacing x by $l - \tau$, we obtain

$$f_1'(\tau - l) = \tfrac{1}{2}[\psi(l - \tau) + \varphi'(l - \tau)] \qquad (0 \leqslant \tau \leqslant l) \qquad (1.12)$$

Here and below the prime denotes differentiation with respect to τ.

Differentiating (1.6) with respect to τ, we express $f_1'(\tau - l)$ by $f_2'(\tau + l)$; Equation (1.12) is then written as

$$f_2'(\tau + l) = -\tfrac{1}{2}[\psi(l - \tau) + \varphi'(l - \tau)]$$

or finally

$$f_2'(\tau) = -\tfrac{1}{2}[\varphi'(2l - \tau) + \psi(2l - \tau)] \qquad (l \leqslant \tau \leqslant 2l) \qquad (1.13)$$

Formulas (1.10) and (1.13) define $f_2'(\tau)$ for $\tau \in [0, 2l]$. Integration determines $f_2(\tau)$ to within a constant. The basic difficulty arises here: this constant cannot be determined without considering simultaneously the problem of finding the function $f_1(\tau)$. The only finite relationship containing only f_2 which can be derived from the condition (1.6) and the first condition (1.9)

$$f_2(x) - f_2(2l - x) = \varphi(x) \qquad (0 \leqslant x \leqslant l) \qquad (1.14)$$

does not determine the constant of integration. In the given example, this difficulty is caused by the fact that the conditions (1.2), (1.3) do not contain a sufficient number of finite relationships between the functions u and v. If such a relationship existed, suppose, instead of the second condition (1.2), then there would be no need to determine the constant of integration, since the function $f_2(\tau)$ for $\tau \in [0, 2l]$ could be given directly.

The question can be answered in a different way. Let us consider an auxiliary problem differing from that formulated above by the fact that the value of $u(x, \tau)$ for $x = 0$ is assumed a certain function of time denoted by $v(\tau)$. The solution of such a problem is well known: it is formed by components depending separately on the initial and the boundary conditions. The initial conditions generate terms in the solution of the form

$$u^{(1)}(x, \tau) = \frac{\Phi(x-\tau) + \Phi(x+\tau)}{2} + \frac{1}{2} \int_{x-\tau}^{x+\tau} \Psi(\xi)\, d\xi \tag{1.15}$$

Here Φ and Ψ are functions obtained by odd continuation with respect to the functions φ and ψ relative to the points $x = 0$ and $x = l$. The terms depending on the boundary conditions are described by the expression

$$u^{(2)}(x, \tau) = \sum_{n=0}^{\infty} \{v[\tau - (2nl + x)] - v[\tau - 2(n+1)l + x]\} \tag{1.16}$$

The notation $v(x)$ has the following meaning:

$$v(x) = \begin{cases} V(x) & (x \geqslant 0) \\ 0 & (x < 0) \end{cases} \tag{1.17}$$

The series in (1.16), therefore, contains for each finite τ a finite number of terms. The general solution is given by the sum of (1.16) and (1.17)

$$u(x, \tau) = u^{(1)}(x, \tau) + u^{(2)}(x, \tau) \tag{1.18}$$

Differentiating (1.18) with respect to x and letting $x = 0$ we obtain

$$u_x(0, \tau) = -V'(\tau) - 2\sum_{n=1}^{\infty} v'(\tau - 2nl) + \Phi'(\tau) + \Psi(\tau) \tag{1.19}$$

Recalling the boundary condition (1.3), we arrive at the nonlinear differential-difference equation

$$V'(\tau) + 2\sum_{n=1}^{\infty} v'(\tau - 2nl) + F[V(\tau)] = \Phi'(\tau) + \Psi(\tau) \tag{1.20}$$

which is integrable for the initial condition $V(0) = \varphi(0)$.

Obviously this procedure is not based on the D'Alembert representation of the general solution for the original equation. Thus, for the one-dimensional wave equation

$$u_{xx} - u_{\tau\tau} + c^2 u = 0 \tag{1.21}$$

under conditions (1.2), (1.3) in place of Equation (1.20), we obtain the integral-differential equation of the type

$$V'(\tau) + 2\sum_{n=1}^{\infty} v'(\tau - 2nl) - c\int_0^{\tau} V(\tau - \theta)\frac{I_1(c\theta)}{\theta}\,d\theta -$$

$$-2c\sum_{n=1}^{\infty}\int_{2nl}^{\tau} v(\tau - \theta)\frac{I_1(c\sqrt{\theta^2 - 4n^2l^2})}{\sqrt{\theta^2 - 4n^2l^2}}\,d\theta + F[V(\tau)] = \Phi'(\tau) + \Psi(\tau) \tag{1.22}$$

2. Let us consider the oscillations in a system consisting of a flat channel of length l with a shut-off valve at the end $x = l$. The valve mass will be denoted by M; let the valve be supported by the spring with a nonlinear characteristic $F = -Cy + Dy^3 - Fy^5$. The spring is placed into a medium the resistance of which is proportional to the first power of velocity. In the stationary condition a fluid flows along the channel under a pressure of p_0 applied at the end $x = 0$. The hydrodynamic quantities referred to the stationary condition will be denoted by the index 0.

It is assumed that the fluid is slightly compressible while its flow velocity is small compared to the velocity of sound. Considered are small deviations from the stationary condition caused by the sudden change in the pressure at the end $x = 0$ of the channel. These deviations will be denoted by v, p, ρ (velocity, pressure, density).

Introducing the nondimensional variables

$$u = \frac{v - v_0}{v_0}, \qquad q = \frac{p - p_0}{p_0} \tag{2.1}$$

it is not difficult to obtain the equations

$$u_\tau = -\lambda q_\xi, \qquad u_\xi = -\lambda q_\tau \tag{2.2}$$

as consequence of the hydrodynamic equations for the condition that the perturbations of all quantities are small compared to their stationary values, and neglecting the products of small quantities compared to the linear terms; while at the same time introducing the notation

$$\tau = \frac{ct}{l}, \qquad \xi = \frac{x}{l}, \qquad \lambda = \frac{p_0}{\rho_0 c v_0} \tag{2.3}$$

(c is the velocity of sound in the unperturbed motion of the fluid).

The boundary conditions will be given in the form [4]

$$q(0, \tau) = -\frac{1}{\lambda}\psi(\tau) \tag{2.4}$$

Here ψ is a given function. For $x = l$ the boundary condition is determined by the valve motion. If y denotes valve displacement (spring) from neutral position of the spring, then the equation of motion for the valve is of the form

$$M\frac{d^2y}{dt^2} + k\frac{dy}{dt} + Cy - Dy^3 + Fy^5 = \Omega p(l, t) \tag{2.5}$$

Here Ω is the cross-sectional area of the valve, k is the damping coefficient.

Let us utilize the stationary equilibrium condition of the valve

$$\Omega p_0 = Cy_0 - Dy_0^3 + Fy_0^5$$

and introduce the nondimensional variable η and the parameters

$$\eta = \frac{y - y_0}{y_0}, \quad \sigma^2 = \frac{Mc^2}{l^2}\frac{y_0}{\Omega p_0}, \quad n = k\frac{c}{l}\frac{y_0}{\Omega p_0}, \quad \Omega p_0\delta_1 = Cy_0 - 3Dy_0^3 + 5Fy_0^5$$

$$-\Omega p_0\delta_2 = 3Dy_0^3 - 10Fy_0^5, \qquad -\Omega p_0\delta_3 = Dy_0^3 - 10Fy_0^5 \tag{2.6}$$

$$\Omega p_0\delta_4 = 5Fy_0^5, \qquad \Omega p_0\delta_5 = Fy_0^5$$

Let us write (2.5) in the following form (dot denotes differentiation with respect to τ):

$$\sigma^2\ddot{\eta} + n\dot{\eta} + \delta_1\eta + \delta_2\eta^2 + \delta_3\eta^3 + \delta_4\eta^4 + \delta_5\eta^5 = q \tag{2.7}$$

We relate η with the variables $u(1, \tau)$ and $q(1, \tau)$. The equation of fluid flow through the valve parts is

$$\Omega v = \alpha\beta y\sqrt{\frac{2p}{\rho}} \tag{2.8}$$

Here α is the discharge coefficient, β is the port width. Equation (2.8) is put in the form

$$\Omega v_0(1 + u) = \alpha\beta y_0(1 + \eta)\sqrt{\frac{2p_0}{\rho_0}\frac{\rho_0}{\rho_0 + \rho}(1 + q)}$$

or using the stationary form of Equation (2.8) and the smallness of the quantities u and q

$$\eta = u(1, \tau) - \mu q(1, \tau) \qquad \left(\mu = \frac{1-E}{2}, \ E = \frac{p_0}{c^2 \rho_0}\right) \tag{2.9}$$

Note that $\mu > 0$, since the equation of state is of the form

$$p_0 = \mathrm{const}\, \rho_0^{\varkappa} \qquad (\varkappa > 1) \tag{2.10}$$

For simplicity, the initial conditions will be given in the form:

for fluid

$$u(\xi, 0) = 0, \qquad q(\xi, 0) = 0 \tag{2.11}$$

for valve

$$\eta(0) = 0, \qquad \dot{\eta}(0) = 0 \tag{2.12}$$

The system of Equations (2.2) is satisfied by the expressions

$$u = -F(\tau - \xi) + f(\tau + \xi), \qquad -\lambda q = F(\tau - \xi) + f(\tau + \xi) \tag{2.13}$$

for arbitrary functions F and f; the form of these functions is determined by additional conditions. As is shown by the condition (2.4)

$$F(\tau) + f(\tau) = \psi(\tau), \qquad \tau > 0 \tag{2.14}$$

Using this relationship we write Equations (2.13) in the form

$$u = -F(\tau - \xi) - F(\tau + \xi) + \psi(\tau + \xi) \tag{2.15}$$

$$-\lambda q = F(\tau - \xi) - F(\tau + \xi) + \psi(\tau + \xi) \tag{2.16}$$

The variable $\eta(\tau)$ will be expressed according to these formulas and (2.9) by

$$\eta = (\nu - 1) F(\tau - 1) - (\nu + 1) F(\tau + 1) + (\nu + 1)\psi(\tau + 1) \equiv$$
$$\equiv -\lambda\nu q(1, \tau) + u(1, \tau) \tag{2.17}$$

where

$$\nu = \frac{\mu}{\lambda} = \frac{1 - E}{2\lambda} \tag{2.18}$$

The initial conditions (2.11), along with (2.14), allow one to determine the function $F(\tau)$ for $-1 \leqslant \tau \leqslant 1$. We have

$$F(\tau)=\begin{cases}0 & (-1\leqslant\tau<0)\\ \psi(\tau) & (\ \ 0<\tau\leqslant 1)\end{cases} \tag{2.19}$$

Substituting (2.17) into (2.7) we arrive at the differential-difference equation of second order and neutral type relative to the function $F(\tau)$. The fact that it was possible to determine $F(\tau)$ in the initial interval of τ values using the boundary condition (2.4) and the initial conditions (2.11) was decisive for the possibility of present application of the methods of [1-3].

Let us restrict ourselves, for simplicity, to the case of $M = 0$, $k = 0$. Then the boundary condition for the valve becomes

$$\delta_1\eta+\delta_2\eta^2+\delta_3\eta^3+\delta_4\eta^4+\delta_5\eta^5=q(1,\tau) \tag{2.20}$$

while the initial conditions (2.12) are omitted.

Substituting Expression (2.17) for η into (2.20) and using (2.16) for determination of the function $F(\tau)$, we obtain the functional equation

$$q=\sum_{s=1}^{5}\delta_s[-\lambda\nu q+u]^s \tag{2.21}$$

For brevity here and below, we use q instead of $q(1,\ \tau)$ and similarly for u. Let us introduce the variables θ, ξ by means of the relations

$$\theta=-\lambda q\cos\varphi+u\sin\varphi,\qquad \zeta=-\lambda q\sin\varphi-u\cos\varphi \qquad (\cot\varphi=\nu) \tag{2.22}$$

Then Equation (2.21) can be easily reduced to

$$\zeta=\sum_{s=1}^{5}\varepsilon_s\theta^s,\qquad \varepsilon_1=-\lambda\nu_1{}^2\delta_1-\nu,\qquad \varepsilon_s=-\lambda\nu_1{}^{1+s}\delta_s \tag{2.23}$$

$$(\nu_1=\sqrt{1+\nu^2},\ s=2,3,4,5)$$

We will assume that

$$\psi(\tau)=\begin{cases}0 & (\tau<0)\\ a=\text{const} & (\tau>0)\end{cases}$$

It is convenient to represent on a single drawing the coordinate axes $-\lambda q$, $-u$, $F(\tau-1)$, $F(\tau+1)$ (Fig. 1). We will show the axes ζ and θ on the same figure. Their orientation relative to the previous axes is, apparently, determined by the value of the parameter ν. Let us draw the curve of the function (2.23) in Fig. 1. With respect to the coordinate system $F(\tau-1)$, $F(\tau+1)$ this curve is a characteristic curve of the functional equation (2.21). The solution of this equation can be effected by the graphical method of Lemerey [1-3]. We will seek a periodic solution of period 2. The corresponding closed polygon is represented by a

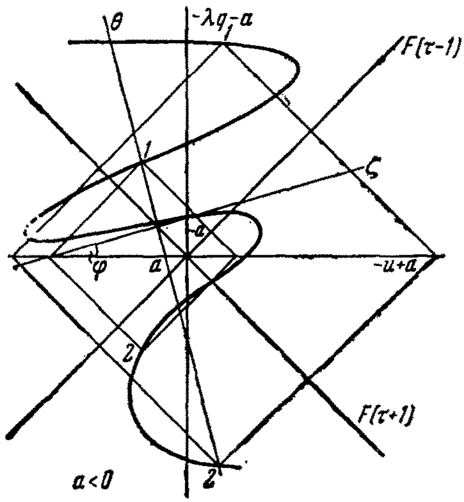

Fig. 1.

square, the two opposite peaks of which rest on the characteristic curve, and the other two on the straight line $F(\tau - 1) = F(\tau + 1)$. The following relationships must be fulfilled for the coordinates of the square peaks 1 and 2 lying on the characteristic curve

$$(\theta_1 + \theta_2)\cos\varphi + (\zeta_1 + \zeta_2)\sin\varphi = a, \quad (\theta_1 - \theta_2)\sin\varphi - (\zeta_1 - \zeta_2)\cos\varphi = 0 \qquad (2.24)$$

We add here Equation (2.23) for the points (ζ_1, θ_1) and (ζ_2, θ_2)

$$\zeta_i = \sum_{s=1}^{5} \varepsilon_s \theta_i^s \qquad (i = 1, 2) \qquad (2.25)$$

Equations (2.24) to (2.25) form a system of four equations with four unknowns ζ_1, θ_1, ζ_2, θ_2. These equations can be easily solved for small φ, when in the first approximation ($\varphi = 0$) we have

$$\theta_1 = -\theta_2 + a, \quad \zeta_1 = \zeta_2 \quad \text{or} \quad \sum_{s=1}^{5} \varepsilon_s \theta_1^s = \sum_{s=1}^{5} \varepsilon_s (a - \theta_1)^s \qquad (2.26)$$

We note immediately the root $\theta_1 = 1/2a$. It is rejected, since it corresponds to the equality $\theta_1 = \theta_2 = 1/2a$, i.e. the coincidence of the peaks 1 and 2. The remaining four roots, as can be easily verified, are determined from the biquadratic equation

$$\varepsilon_5\theta^4 - \gamma_1\theta^2 - \gamma_2 = 0 \qquad (2.27)$$

Here

$$\gamma_1 = -10\frac{a^2}{2^2}\varepsilon_5 - 2a\varepsilon_4 - \varepsilon_3, \quad \gamma_2 = -5\frac{a^4}{2^4}\varepsilon_5 - 4\frac{a^3}{2^3}\varepsilon_4 - 3\frac{a^2}{2^2}\varepsilon_3 - a\varepsilon_2 - \varepsilon_1 \tag{2.28}$$

From (2.27) we obtain

$$(\theta^2)_{1,2} = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 + 4\varepsilon_5\gamma_2}}{2\varepsilon_5}$$

Let us assume for simplicity that the order of smallness of ε_s compared to unity is equal to $s - 1$; let ε_1 be of the order of unity. Then for θ^2 we obtain approximately two values

$$(\theta^2)_1 = \frac{\gamma_1}{\varepsilon_5} + \frac{\gamma_2}{\gamma_1}, \qquad (\theta^2)_2 = -\frac{\gamma_2}{\gamma_1}$$

The assumption regarding the order of smallness of ε_s permits one to consider $\gamma_1 < 0$, $\gamma_2 > 0$. Therefore

$$\theta_{11} = \sqrt{\frac{\gamma_1}{\varepsilon_5} + \frac{\gamma_2}{\gamma_1}}, \qquad \theta_{21} = -\sqrt{\frac{\gamma_1}{\varepsilon_5} + \frac{\gamma_2}{\gamma_1}} \tag{2.29}$$

$$\theta_{12} = \sqrt{-\frac{\gamma_2}{\gamma_1}}, \qquad \theta_{22} = -\sqrt{-\frac{\gamma_2}{\gamma_1}} \tag{2.30}$$

We thus obtain two limit cycles. They are shown in Fig. 1. The smallness of φ is connected with the large value of the parameter ν.

In the second limiting case of small ν (φ close to $\pi/2$) we obtain in first approximation ($\varphi = 1/2\ \pi$)

$$\zeta_1 = a - \zeta_2, \qquad \theta_1 = \theta_2$$

Equations (2.25) show that the given requirements cannot be satisfied; i.e. for $\varphi \approx \pi/2$ there exists no solution with period 2. This statement is illustrated in Fig. 2 where the form of the characteristic curve is shown for $\varphi \approx \pi/2$.

The parameter ν is expressed through the quantities characterizing the stationary condition of the system as follows:

$$\nu = \frac{1 - E}{2\lambda} = \frac{1 - \varkappa^{-1}}{2c}\varkappa v_0 = \frac{\varkappa - 1}{2}\frac{v_0}{c} \tag{2.31}$$

It is supposed that $v_0 < c$, therefore sufficiently large values of $\kappa = c_p/c_v$ correspond to the large values of ν.

Let us pass to the stability investigation of the periodic solutions obtained. For this purpose we compute the relation $|dF(\tau - 1)/dF(\tau + 3)|$ for the limit motion; if this relation is larger than unity stability takes place, if not then instability exists.

Let us differentiate (2.21) with respect to $F(\tau + 3)$; using Formulas (2.15) and (2.16) for $\xi = 1$ we obtain

$$\left[\frac{dF(\tau-1)}{dF(\tau+3)} - \frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} = -\frac{1}{\lambda}\left(\frac{\partial\Phi}{\partial q}\right)_{\lim}\left[\frac{dF(\tau-1)}{dF(\tau+3)} - \frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} - \\ - \left(\frac{\partial\Phi}{\partial u}\right)_{\lim}\left[\frac{dF(\tau-1)}{dF(\tau+3)} + \frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} \tag{2.32}$$

The right-hand side of (2.21), multiplied by $(-\lambda)$, is denoted by Φ, with all terms in (2.32) included for the limit periodical motion. The derivative

$$[dF(\tau+1)/dF(\tau+3)]_{\lim}$$

can be eliminated from (2.32) by means of the equation

$$\left\{\left[\frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} - 1\right\} = -\frac{1}{\lambda}\left(\frac{\partial\Phi}{\partial q}\right)_{\lim}\left\{\left[\frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} - 1\right\} - \\ - \left(\frac{\partial\Phi}{\partial u}\right)_{\lim}\left\{\left[\frac{dF(\tau+1)}{dF(\tau+3)}\right]_{\lim} + 1\right\} \tag{2.33}$$

also originating from (2.21). From (2.32) and (2.33) it follows that

$$\left|\frac{dF(\tau-1)}{dF(\tau+3)}\right|_{\lim} = \frac{(1+\lambda^{-1}\,\partial\Phi/\partial q - \partial\Phi/\partial u)^2_{\lim}}{(1+\lambda^{-1}\,\partial\Phi/\partial q + \partial\Phi/\partial u)^2_{\lim}} \tag{2.34}$$

The following relations are valid

$$\frac{\partial\Phi}{\partial u}\Big|_{\lim} = \frac{\partial\Phi}{\partial\theta}\Big|_{\lim}\frac{\partial\theta}{\partial u} = \sin\varphi\left(\frac{\partial\Phi}{\partial\theta}\right)_{\lim}$$
$$\frac{\partial\Phi}{\partial q}\Big|_{\lim} = \frac{\partial\Phi}{\partial\theta}\Big|_{\lim}\frac{\partial\theta}{\partial q} = -\lambda\cos\varphi\left(\frac{\partial\Phi}{\partial\theta}\right)_{\lim}$$

Hence

$$\left|\frac{dF(\tau-1)}{dF(\tau+3)}\right|_{\lim} = \frac{[1-(\partial\Phi/\partial\theta)_{\lim}(\cos\varphi+\sin\varphi)]^2}{[1-(\partial\Phi/\partial\theta)_{\lim}(\cos\varphi-\sin\varphi)]^2}$$

For the case of small $\varphi > 0$, the stability according to Koenigs apparently takes place for $[\partial\Phi/\partial\theta]_{\lim} < 0$ or

$$\sum_{s=1}^{5} s\delta_s v_1{}^s\theta^{s-1} > 0$$

The condition obtained with respect to the order of magnitude of δ_s allows one to state that the latter inequality can be satisfied by the

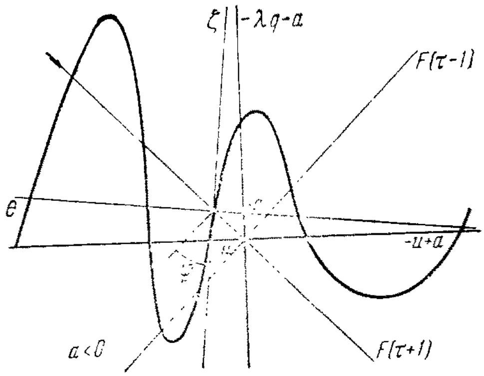

Fig. 2.

values of θ_{11} and θ_{21} (see (2.29)). The roots of θ_{12} and θ_{22} yield an opposite inequality. Thus, under these conditions the "outer" (Fig. 1) limit cycle is stable while the inner one is unstable.

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